SOME APPLICATIONS OF DEGENERATE POLY-BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we consider degenerate poly-Bernoulli numbers and polynomials associated with polylogarithmic function and p-adic invariant integral on $\mathbb{Z}_p$. By using umbral calculus, we derive some identities of those numbers and polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. For $k \in \mathbb{Z}$, the polylogarithmic function $\mathrm{Li}_k(x)$ is defined by $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$. For $k = 1$, we have $\mathrm{Li}_1(x) = -\log(1-x)$.

In [4], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.1}$$

Note that $\lim_{\lambda\to 0}\beta_{n,\lambda}(x) = B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials. When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

It is known that the poly-Bernoulli polynomials are defined by the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \quad (\text{see [8]}). \tag{1.2}$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the poly-Bernoulli numbers.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\,\mu_0\left(x+p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \quad (\text{see [13]}).\end{aligned} \tag{1.3}$$

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From (1.3), we have

$$\int_{\mathbb{Z}_p} f(x+1)\,d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = f'(0), \tag{1.4}$$

where $f'(0) = \frac{df(x)}{dx}\Big|_{x=0}$ (see [1–17]).

By (1.4), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+\lambda t)^{(x+y)/\lambda}\,d\mu_0(y) &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{\log(1+\lambda t)}{\lambda t}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}D_{n-l}\beta_{l,\lambda}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{1.5}$$

where D_n are the Daehee numbers of the first kind given by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad (\text{see [9]}). \tag{1.6}$$

Let $\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \Big| a_k \in \mathbb{C}_p \right\}$ be the algebra of formal power series in a single variable t. Let $\mathbb{P}$ be the algebra of polynomials in a single vairable x over $\mathbb{C}_p$. We denote the action of the linear functional $L \in \mathbb{P}^*$ on a polynomial $p(x)$ by $\langle L|\, p(x)\rangle$, which is linearly extended as $\langle cL + c'L'|\, p(x)\rangle = c\langle L|\, p(x)\rangle + c'\langle L'|\, p(x)\rangle$, where $c, c' \in \mathbb{C}_p$. We define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|\, x^n\rangle = a_n, \quad \text{for all } n \geq 0 \text{ and } f(t) \in \mathcal{F}. \tag{1.7}$$

By (1.7), we easily get

$$\left\langle t^k \middle|\, x^n \right\rangle = n!\delta_{n,k}, \quad (n, k \geq 0), \tag{1.8}$$

where $\delta_{n,k}$ is the Kronecker's symbol (see [15]).

For $f_L(t) = \sum_{k=0}^{\infty} \left\langle L|\, x^k\right\rangle \frac{t^k}{k!}$, we have $\langle f_L(t)|\, x^n\rangle = \langle L|\, x^n\rangle$. The map $L \mapsto f_L(t)$ is vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra.

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [10, 15]). If $o(f(t)) = 1$ (respectively, $o(f(t)) = 0$), then $f(t)$ is called a delta (respectively, an invertible) series.

For $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\left\langle g(t) f(t)^k \middle|\, s_n(x) \right\rangle = n!\delta_{n,k}(n, k \geq 0)$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [15]).

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, by (1.8), we get

$$\left\langle e^{yt} \middle|\, p(x)\right\rangle = p(y), \quad \langle f(t) g(t)|\, p(x)\rangle = \langle g(t)|\, f(t) p(x)\rangle = \langle f(t)|\, g(t) p(x)\rangle \tag{1.9}$$

and

$$f\left(t\right)=\sum_{k=0}^{\infty}\left\langle f\left(t\right)\middle|x^{k}\right\rangle\frac{t^{k}}{k!},\quad p\left(x\right)=\sum_{k=0}^{\infty}\left\langle t^{k}\middle|p\left(x\right)\right\rangle\frac{x^{k}}{k!},\quad(\text{see [15]}).\tag{1.10}$$

From (1.10), we note that

$$p^{(k)}\left(0\right)=\left\langle t^{k}\middle|p\left(x\right)\right\rangle=\left\langle 1\middle|p^{(k)}\left(x\right)\right\rangle,\quad(k\geq 0),\tag{1.11}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x=0$.

By (1.11), we get

$$t^{k}p\left(x\right)=p^{(k)}\left(x\right)=\frac{d^{k}}{dx^{k}}p\left(x\right),\quad(k\geq 0).\tag{1.12}$$

In [15], it is known that

$$s_{n}\left(x\right)\sim\left(g\left(t\right),f\left(t\right)\right)\iff\frac{1}{g\left(\overline{f}\left(t\right)\right)}e^{x\overline{f}(t)}=\sum_{n=0}^{\infty}s_{n}\left(x\right)\frac{t^{n}}{n!},\quad(x\in\mathbb{C}_{p}),\tag{1.13}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f\left(\overline{f}(t)\right)=\overline{f}\left(f(t)\right)=t$.

From (1.12), we can easily derive the following equation:

$$e^{yt}p\left(x\right)=p\left(x+y\right),\quad\text{where }p\left(x\right)\in\mathbb{P}=\mathbb{C}_{p}\left[x\right].\tag{1.14}$$

In this paper, we study degenerate poly-Bernoulli numbers and polynomials associated with polylogarithm function and p-adic invariant integral on $\mathbb{Z}_p$. Finally, we give some identities of those numbers and polynomials which are derived from umbral calculus.

2. Some applications of degenerate poly-Bernoulli numbers

Now, we consider the degenerate poly-Bernoulli polynomials which are given by the generating function

$$\frac{\mathrm{Li}_{k}\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}\left(x\right)\frac{t^{n}}{n!},\quad(k\in\mathbb{Z}).\tag{2.1}$$

From (1.13) and (2.1), we have

$$\beta_{n,\lambda}^{(k)}\left(x\right)\sim\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\mathrm{Li}_{k}\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)},t\right),\tag{2.2}$$

and

$$\beta_{n,\lambda}^{(k)}\left(x\right)=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(k)}x^{n-l},\tag{2.3}$$

where $\beta_{l,\lambda}^{(k)}=\beta_{l,\lambda}^{(k)}(0)$ are called the degenerate poly-Bernoulli numbers.

Thus, by (2.3), we get

$$\begin{aligned}\int_{x}^{x+y}\beta_{n,\lambda}^{(k)}\left(u\right)du&=\frac{1}{n+1}\left\{\beta_{n+1,\lambda}^{(k)}\left(x+y\right)-\beta_{n+1,\lambda}^{(k)}\left(x\right)\right\}\\&=\frac{e^{yt}-1}{t}\beta_{n,\lambda}^{(k)}\left(x\right).\end{aligned}\tag{2.4}$$

Let $f(t)$ be the linear functional such that

$$\langle f(t)|\, p(x)\rangle = \int_{\mathbb{Z}_p} \frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} p(x)\, d\mu_0(x)$$

for all polynomials $p(x)$. Then it can be determined as follows: for any $p(x) \in \mathbb{P}$,

$$\left\langle \frac{t}{e^t-1} \middle| p(x) \right\rangle = \int_{\mathbb{Z}_p} p(x) d\mu_0(x).$$

Replacing $p(x)$ by $\frac{e^t-1}{t} h(t)p(x)$, for $h(t) \in \mathcal{F}$, we get

$$\langle h(t)|\, p(x)\rangle = \int_{\mathbb{Z}_p} \frac{e^t-1}{t} h(t)p(x) d\mu_0(x). \tag{2.5}$$

In particular, for $h(t)=1$, we obtain

$$\int_{\mathbb{Z}_p} \frac{e^t-1}{t} p(x) d\mu_0(x) = p(0). \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem as a special case.

Theorem 1. *For $p(x) \in \mathbb{P}$, we have*

$$\left\langle \frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \middle| p(x) \right\rangle$$
$$= \int_{\mathbb{Z}_p} \frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} p(x)\, d\mu_0(x),$$

and

$$\left\langle \frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \int_{\mathbb{Z}_p} e^{yt} d\mu_0(y) \middle| p(x) \right\rangle$$
$$= \int_{\mathbb{Z}_p} \frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} p(x)\, d\mu_0(x).$$

In particular,

$$\beta_{n,\lambda}^{(k)} = \left\langle \frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)} \int_{\mathbb{Z}_p} e^{yt} d\mu_0(y) \middle| x^n \right\rangle, \quad (n \geq 0).$$

Note that

$$\left\langle \int_{\mathbb{Z}_p} e^{yt} d\mu_0(y) \middle| \frac{e^t-1}{t} \beta_{n,\lambda}^{(k)}(x) \right\rangle$$
$$= \frac{1}{n+1} \left\langle \frac{t}{e^t-1} \middle| \beta_{n+1,\lambda}^{(k)}(x+1) - \beta_{n+1,\lambda}^{(k)}(x) \right\rangle$$

$$=\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_l\left(\beta_{n+1-l,\lambda}^{(k)}(1)-\beta_{n+1-l,\lambda}^{(k)}\right)=\beta_{n,\lambda}^{(k)}.$$

It is easy to show that

$$\begin{aligned}(2.7)\qquad &\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)^n\,d\mu_0(y)\frac{t^n}{n!}\\ &=\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\times\frac{t}{e^t-1}e^{xt}\\ &=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k)}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, by (2.7), we get

$$\begin{aligned}(2.8)\qquad \beta_{n,\lambda}^{(k)}(x)&=\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\int_{\mathbb{Z}_p}(x+y)^n\,d\mu_0(y)\\ &=\frac{\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}x^n\end{aligned}$$

Therefore, by (2.8), we obtain the following theorem.

Theorem 2. *For $p(x)\in\mathbb{P}$, we have*

$$\begin{aligned}&\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\int_{\mathbb{Z}_p}p(x+y)\,d\mu_0(y)\\ &=\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\int_{\mathbb{Z}_p}e^{yt}p(x)\,d\mu_0(y)\\ &=\frac{\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}p(x).\end{aligned}$$

For $r\in\mathbb{N}$, let us consider the higher-order degenerate poly-Bernoulli polynomials as follows:

(2.9)

$$\begin{aligned}&\left(\frac{(e^t-1)\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r+x)t}d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &=\left(\frac{\operatorname{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k,r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we obtain

$$\beta_{n,\lambda}^{(k,r)}(x)=\left(\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r x^n \tag{2.10}$$
$$=\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r$$
$$\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)^n\,d\mu_0(x_1)\cdots d\mu_0(x_r),$$

where $n\geq 0$.

Here, for $x=0$, $\beta_{n,\lambda}^{(k,r)}=\beta_{n,\lambda}^{(k,r)}(0)$ are called the degenerate poly-Bernoulli numbers of order r. From (2.9), we note that

$$\beta_{n,\lambda}^{(k)}(x)\sim\left(\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}\right)^r,t\right). \tag{2.11}$$

Therefore, by (2.10), we obtain the following theorem.

Theorem 3. *For $p(x)\in\mathbb{P}$ and $r\in\mathbb{N}$, we have*

$$\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}p(x_1+\cdots+x_r+x)\,d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$=\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r)t}p(x)d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$=\left(\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r p(x).$$

Let us consider the linear functional $f_r(t)$ such that

(2.12)

$$\langle f_r(t)|\,p(x)\rangle$$
$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r)$$

for all polynomials $p(x)$. Then it can be determined in the following way: for $p(x)\in\mathbb{P}$,

$$\left\langle\left(\frac{t}{e^t-1}\right)^r\middle|\,p(x)\right\rangle=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r).$$

Replacing $p(x)$ by $\left(\frac{e^t-1}{t}h(t)\right)^r p(x)$, for $h(t)\in\mathcal{F}$, we have

(2.13)
$$\langle h(t)^r|\,p(x)\rangle=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{e^t-1}{t}h(t)\right)^r p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r).$$

In particular, for $h(t)=1$, we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{e^t-1}{t}\right)^r p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r)=p(0). \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 4. *For $p(x)\in\mathbb{P}$, we have*

$$\left\langle\left(\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r\middle|\,p(x)\right\rangle$$
$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r),$$

and

$$\left\langle\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r)t}d\mu_0(x_1)\cdots d\mu_0(x_r)\middle|\,p(x)\right\rangle$$
$$=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r p(x)|_{x=x_1+\cdots+x_r}\,d\mu_0(x_1)\cdots d\mu_0(x_r).$$

In particular,

$$\beta_{n,\lambda}^{(k,r)}=\left\langle\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r)t}d\mu_0(x_1)\cdots d\mu_0(x_r)\middle|\,x^n\right\rangle.$$

Remark. It is not difficult to show that

$$\left\langle\left(\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\right)^r\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{(x_1+\cdots+x_r)t}d\mu_0(x_1)\cdots d\mu_0(x_r)\middle|\,x^n\right\rangle$$
$$=\sum_{n=n_1+\cdots+n_r}\binom{n}{n_1,\ldots,n_r}\left\langle\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\int_{\mathbb{Z}_p}e^{x_{n_1}t}d\mu_0(x_1)\middle|\,x^{m_1}\right\rangle\times\cdots$$
$$\times\left\langle\frac{(e^t-1)\,\mathrm{Li}_k\left(1-(1+\lambda t)^{-\frac{1}{\lambda}}\right)}{t\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)}\int_{\mathbb{Z}_p}e^{x_{n_r}t}d\mu_0(x_{n_r})\middle|\,x^{n_r}\right\rangle.$$

Thus, we get

$$\beta_{n,\lambda}^{(k,r)}=\sum_{n=n_1+\cdots+n_r}\binom{n}{n_1,\ldots,n_r}\beta_{n_1,\lambda}^{(k)}\cdots\beta_{n_r,\lambda}^{(k)}.$$

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